ESSAY

The energy gap in nuclear matter

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The vibrational spectra of nuclei shows an energy gap of about 1 MeV between even-even and odd nuclei in addition to contributions from the pairing interaction to the binding energy of the nucleus. We perform a crude calculation of this energy gap assuming an infinite nucleus and a nonsingular square-well potential fit to low-energy ${}^1S_0$ scattering for the two-body nuclear potential. We rely on BCS theory to derive an equation describing the energy gap, the so called BCS gap equation, which is possible due to the analogous interaction between nucleon pairing and Cooper pairs. We find that the energy gap depends strongly on the nuclear density and effective mass at the Fermi surface. For typical nuclear densities, the gap is found to be small compared with observations. For greater agreement with experiment, a finite nuclei model and numerical methods must be employed.

I. INTRODUCTION

The semiempirical mass formula indicates that the last pair of like particles (pp or nn) in a nucleus contributes an extra amount of energy

$$\delta(A,Z) = \begin{cases} +(34\text{ MeV})A^{-\frac{1}{2}} & Z, N \text{ even } (A \text{ even}) \\ 0 & A \text{ odd} \\ -(34\text{ MeV})A^{-\frac{1}{2}} & Z, N \text{ odd } (A \text{ even}) \end{cases} \tag{1}$$

to the binding energy of nuclei where Z is the atomic or proton number, N is the neutron number, and A is the mass number [1]. This is known as the pairing term or pairing interaction and it encapsulates the effect of spin coupling. It depends on the evenness or oddness of both Z and the neutron number N, and consequently A.

In addition to this contribution, experimental observations reviewed by Alder et al. [2] in 1956 indicate an energy gap of approximately 1 MeV in the vibrational spectrum of heavy even-even nuclei (i.e. even-N, even-Z nuclei) as shown in Fig. 1. This gap is greater for lighter nuclei.

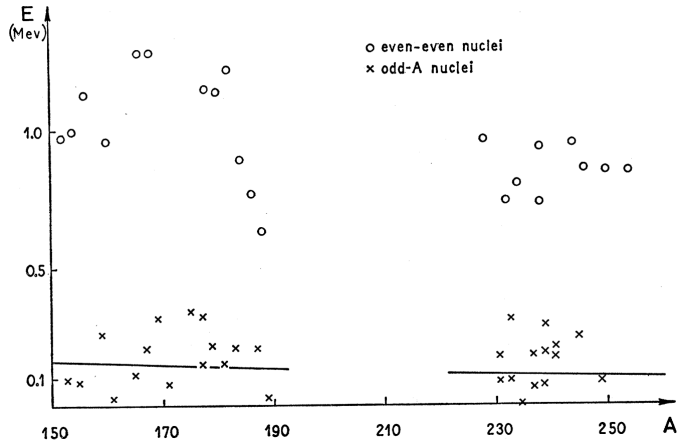

FIG. 1. Energies of first vibrationally excited states in deformed nonspherical nuclei. The plot has been taken from Bohr, Mottelson, and Pines [3] who in turn obtained the experimental data from *Nuclear Data Cards* [National Research Council, Washington, D. C]. The solid line gives the binding energy due to the pairing interaction. It is interesting to note that at the time of publication, 1958, it was believed that the contribution of the pairing term was $(50\text{ MeV})A^{-1}$ which we now know more accurately as described by (1). The figure contains all the available data at the time for nuclei with $150 < A < 190$ and $228 < A$. In these regions the nuclei possess a nonspherical equilibrium shape otherwise vibrationally excited states would not exhibit a gap. Alder et al. [2] reviews the effects of spherical asymmetry on the energy levels of nuclei, suggesting that it may be responsible for this energy gap.

A couple of years later in 1958, Bohr, Mottelson, and Pines [3] suggested that the observed energy gap may be

analogous to the one observed in the electronic excitation of a superconducting metal as the low-lying states of atomic nuclei might have a collective character which could be described in a manner similar to Cooper pairs in BCS theory. It is interesting to note that their work was submitted to *Phys. Rev.* only a month after Bardeen, Cooper, and Schrieffer's famous paper [4] was published, proposing the first complete theory of microscopic superconductivity. However, Pines was very familiar with their work having been Bardeen's first post-doctoral fellow in the early 1950's which he discusses in an interesting autobiographical article [5] just recently published.

In 1960, Emery and Sessler [6] provided a theoretical calculation for the energy gap in nuclear matter assuming an infinite nucleus and employing the Gammel-Thaler potential [7] and numerical methods. The potential was chosen as it fits two-body scattering data up to a few hundred MeV and has been successful in reproducing the bulk properties of nuclear matter. They find that the energy gap it is a very strong function of the nuclear density and the effective mass at the Fermi surface. Emery and Sessler noted that their calculations predicted a smaller gap than observations suggested. This was due to BCS theory being only applicable, at least at the time, to infinite systems of strongly interacting fermions and thus they had to treat the nucleus as infinite, ignoring surface effects and Coulomb interactions. Qualitatively similar results have been obtained by Brueckner et al. [8] who used the nuclear matter two-body K-matrix instead of the bare two-body potential in the BCS gap equation. They also calculated a smaller gap than expected suggesting that the choice of potential may not be relevant. Instead it seems that either the energy gap is a feature of finite nuclear matter or that BCS theory may be not applicable to nuclear matter.

In 1963, Kennedy, Wilets, and Henley published a pair of papers [9, 10] extensively describing the theory of the nuclear energy gap and detailing theoretical progress to describe it. They also suggested explanations for the underestimation of the gap energy, one of them being that the gap emerges as a surface property, which would have been ignored in an infinite model of the nucleus. It is extremely difficult to make a quantitative evaluation of the energy gap for finite nuclei as realistic nucleon-nucleon forces must be used and so an infinite nucleus model was used, which proved fruitful in previous calculations [2]. While infinite models were not satisfactory quantitatively, they provided the correct qualitative results and insight into the energy gap.

Attempts at numerical calculations have been made for finite nuclei models, achieving better agreement with experiment. In the mid 1960's Kennedy, Wilets, and Henley [11, 12] successfully used a slab model which confined the nucleus in one dimension by an infinite square well (keeping two others infinite in extent) thus giving the nucleus a surface for the energy gap to arise from. Other attempts include Thompson's cubical model [13], and Madland and Nix's [14] use of a mean-field theory approximation (replacing operators with their average values for small fluctuations). These refinements brought the theory in much better agreement with experiment. There does not seem to be work more recent than the mid 1960's done on the subject, which makes sense seeing that this energy gap was probably never much more than a curiosity in the excitation spectra of nuclei. An exciting analogy was made right when the BCS theory was proposed and further work quickly explained the gap well enough such that no further work was motivated.

In this calculation we shall use a simple nonsingular square-well potential fit to low-energy $^1S_0$ scattering for the two-body nuclear potential allowing us to perform the calculation analytically. While our results will greatly underestimate the gap, we can learn much of its nature from our calculation. There is also much to be learned from the application of BCS theory to nuclear matter, making this an interesting problem to study from the prospective of a student despite it being insignificant today as evidenced by the lack of published material on the subject over the past five decades or so.

II. DERIVATION OF THE BCS GAP EQUATION

In 1956 Cooper [15] showed that a degenerate Fermi gas is unstable and tends to favor the formation of bound pairs for arbitrarily weak attractive interactions. Such bound states of electrons are now called Cooper pairs. This was a major step towards a complete theory of superconductivity as the existence of bound states may be used to explain the empirically observed energy gap in the spectrum of electronic excitations of a metallic superconductor. The idea was the in order to excite an electron from below the Fermi surface to above, the Cooper pair bound state must be broken. The binding energy of the Cooper pair thus provides the energy gap.

It was suspected that such a theory, meant to describe superconductivity, could also explain certain phenomena in nuclear physics. This problem of the energy gap in nuclear matter in one of them. After all, nucleons pair up similarly to Cooper pairs ($n\uparrow$ with $n\downarrow$ and $p\uparrow$ with $p\downarrow$) throughout the nucleus and if the nucleus was treated as a degenerate Fermi gas, the full mathematics of BCS theory may be used to tackle problems in nuclear physics. It is

worth noting that the binding energy of a Cooper pair is on the order of 10^{-3} eV while we are seeking to calculate an energy gap of 1 MeV, nine orders of magnitude greater. Of course, the energy scales at play inside the nucleus are naturally much greater.

We shall first introduce the fermionic creation operator $c^\dagger_{\mathbf{k}\sigma}$ which creates an electron of momentum $\mathbf{k}$ and spin σ, and its corresponding annihilation operator $c_{\mathbf{k}\sigma}$. These operators obey the usual anticommutation relations for fermions

$$\{c_{\mathbf{k}\sigma}, c^\dagger_{\mathbf{k}'\sigma'}\} = \delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\sigma\sigma'} \tag{2}$$

$$\{c_{\mathbf{k}\sigma}, c_{\mathbf{k}'\sigma'}\} = \{c^\dagger_{\mathbf{k}\sigma}, c^\dagger_{\mathbf{k}'\sigma'}\} = 0 \tag{3}$$

We can define a particle number operator in the standard way

$$n_{\mathbf{k}\sigma} = c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \tag{4}$$

as the observable which counts the number of particles with momentum $\mathbf{k}$ and spin σ. Its eigenstates are the Fock states, that is, states of fixed particle number.

We now have to start our derivation somewhere. We shall roughly follow the derivation given by Yoonseok Lee in his lecture notes on "BCS Theory and Superconductivity" [16]. Let us start with the so-called pairing (or BCS) Hamiltonian

$$H = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} n_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{l}} V_{\mathbf{k}\mathbf{l}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow} \tag{5}$$

where $V_{\mathbf{k}\mathbf{l}} = \langle \mathbf{k}|V|\mathbf{l}\rangle$. The first term is the usual kinetic energy $\epsilon_{\mathbf{k}}$ of the electrons summed over all electrons of momenta $\mathbf{k}$ and spin $\sigma \in \{\uparrow, \downarrow\}$. The second term is the interaction term between all Cooper pairs $\mathbf{k}\uparrow, -\mathbf{k}\downarrow$ and $\mathbf{l}\uparrow, -\mathbf{l}\downarrow$. There are other interaction terms, all normal-ordered products of four creation and annihilation operators as this is an example of a four-fermion interaction, however Cooper pairs cannot bind in any other way and so these terms are not included. The matrix elements $V_{\mathbf{k}\mathbf{l}}$ are left as general as possible for now until we switch gears to studying nuclear matter.

We can then add a $-\mu N$ term to (5) where μ is the chemical potential and N is the number operator, giving

$$H' \equiv H - \mu N = H - \mu \sum_{\mathbf{k},\sigma} n_{\mathbf{k}\sigma} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} n_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{l}} V_{\mathbf{k}\mathbf{l}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow} \tag{6}$$

where the new energies $\xi_{\mathbf{k}}$ are now measured from the top of the Fermi sea

$$\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu = \frac{\hbar^2 k^2}{2m} - \mu \tag{7}$$

This is mathematically equivalent to redefining the zero of kinetic energy to be at $\mu = E_F(T=0)$ where E_F is the Fermi energy.

We now define the quantity

$$b_{\mathbf{k}} = c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle \approx 0 \tag{8}$$

where we have applied the mean-field approximation (sometimes called the BCS approximation) so that we may express the product of operators as a fluctuation, which happens to be very small in this case. One way of reasoning this is that in a normal state of matter we would expect no Cooper pairs to form so the operator $c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}$ which creates an electron pair with opposite momenta and spins (i.e. a Cooper pair) should average out to zero. Moving on, this in turn allows us to write the specific product of two fermionic annihilation operator as

$$c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} = b_{\mathbf{k}} + (c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - b_{\mathbf{k}}) \tag{9}$$

Inserting (9) into the second summation of (6), the product of the four operators becomes

$$\begin{aligned} c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}c_{-\mathbf{l}\downarrow}c_{\mathbf{l}\uparrow} &= \left(c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}\right)^\dagger c_{-\mathbf{l}\downarrow}c_{\mathbf{l}\uparrow} \\ &= \left[b_{\mathbf{k}} + \left(c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow} - b_{\mathbf{k}}\right)\right]^\dagger \left[b_{\mathbf{l}} + \left(c_{-\mathbf{l}\downarrow}c_{\mathbf{l}\uparrow} - b_{\mathbf{l}}\right)\right] \\ &= b^\star_{\mathbf{k}}b_{\mathbf{l}} + b^\star_{\mathbf{k}}\left(c_{-\mathbf{l}\downarrow}c_{\mathbf{l}\uparrow} - b_{\mathbf{l}}\right) + \left(c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow} - b^\star_{\mathbf{k}}\right)b_{\mathbf{l}} + \underbrace{\left(c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow} - b_{\mathbf{k}}\right)\left(c_{-\mathbf{l}\downarrow}c_{\mathbf{l}\uparrow} - b_{\mathbf{l}}\right)}_{=0 \text{ by the mean-field approximation}} \\ &= c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}b_{\mathbf{l}} + b^\star_{\mathbf{k}}c_{-\mathbf{l}\uparrow}c_{\mathbf{l}\uparrow} - b^\star_{\mathbf{k}}b_{\mathbf{l}} \end{aligned} \tag{10}$$

and thus we may write (6) as

$$H' = \sum_{\mathbf{k},\sigma}\xi(\mathbf{k})n_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{l}}V_{\mathbf{k}\mathbf{l}}\left(c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}b_{\mathbf{l}} + b^\star_{\mathbf{k}}c_{-\mathbf{l}\uparrow}c_{\mathbf{l}\uparrow} - b^\star_{\mathbf{k}}b_{\mathbf{l}}\right) \tag{11}$$

We can also define the so-called gap energy as

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{l}}V_{\mathbf{k}\mathbf{l}}b_{\mathbf{l}} = -\sum_{\mathbf{l}}V_{\mathbf{k}\mathbf{l}}\left\langle c_{-\mathbf{l}\downarrow}c_{\mathbf{l}\uparrow}\right\rangle \approx -\sum_{\mathbf{l}}V_{\mathbf{k}\mathbf{l}}\left\langle c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}\right\rangle \tag{12}$$

where we used (8) so that $\langle c_{-\mathbf{l}\downarrow}c_{\mathbf{l}\uparrow}\rangle \approx \langle c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}\rangle$. This now lets us write the second summation in (11) as

$$\begin{aligned} &\sum_{\mathbf{k},\mathbf{l}}V_{\mathbf{k}\mathbf{l}}\left(c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}b_{\mathbf{l}} + b^\star_{\mathbf{k}}c_{-\mathbf{l}\uparrow}c_{\mathbf{l}\uparrow} - b^\star_{\mathbf{k}}b_{\mathbf{l}}\right) \\ &= \sum_{\mathbf{k}}\left[\sum_{\mathbf{l}}V_{\mathbf{k}\mathbf{l}}c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}b_{\mathbf{l}} + \sum_{\mathbf{l}}V_{\mathbf{k}\mathbf{l}}b^\star_{\mathbf{k}}c_{-\mathbf{l}\uparrow}c_{\mathbf{l}\uparrow} - \sum_{\mathbf{l}}V_{\mathbf{k}\mathbf{l}}b^\star_{\mathbf{k}}b_{\mathbf{l}}\right] \\ &= \sum_{\mathbf{k}}\left[\underbrace{\sum_{\mathbf{l}}V_{\mathbf{k}\mathbf{l}}b_{\mathbf{l}}}_{=-\Delta_{\mathbf{k}}}c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow} + \underbrace{\sum_{\mathbf{l}}V_{\mathbf{k}\mathbf{l}}b^\star_{\mathbf{l}}}_{=-\Delta^\star_{\mathbf{k}}}c_{-\mathbf{l}\uparrow}c_{\mathbf{l}\uparrow} - \underbrace{\sum_{\mathbf{l}}V_{\mathbf{k}\mathbf{l}}b_{\mathbf{l}}}_{=-\Delta_{\mathbf{k}}}b^\star_{\mathbf{k}}\right] \\ &= -\sum_{\mathbf{k}}\left(\Delta_{\mathbf{k}}c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow} + \Delta^\star_{\mathbf{k}}c_{-\mathbf{l}\uparrow}c_{\mathbf{l}\uparrow} - \Delta_{\mathbf{k}}b^\star_{\mathbf{k}}\right) \end{aligned} \tag{13}$$

where we again used the mean-field approximation (8) so that $b^\star_{\mathbf{k}} = b^\star_{\mathbf{l}}$ between the second and third lines. This further lets us write the Hamiltonian (11) as

$$H' = \sum_{\mathbf{k},\sigma}\xi_{\mathbf{k}}n_{\mathbf{k}\sigma} - \sum_{\mathbf{k}}\left(\Delta_{\mathbf{k}}c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow} + \Delta^\star_{\mathbf{k}}c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow} - \Delta_{\mathbf{k}}b^\star_{\mathbf{k}}\right) \tag{14}$$

We now wish to diagonalize this Hamiltonian which unfortunately has a chunk of interacting terms. This is done by introducing a Bogoliubov-Valatin transformation [17] which expresses the electron annihilation and creation operators in terms of "Bogoliubov quasiparticle" operators

$$c_{\mathbf{k}\uparrow} = u^\star_{\mathbf{k}}\gamma_{\mathbf{k}\uparrow} + v_{\mathbf{k}}\gamma^\dagger_{-\mathbf{k}\downarrow} \tag{15}$$

$$c^\dagger_{-\mathbf{k}\downarrow} = -v^\star_{\mathbf{k}}\gamma_{\mathbf{k}\uparrow} + u_{\mathbf{k}}\gamma^\dagger_{-\mathbf{k}\downarrow} \tag{16}$$

with the condition that $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$. Due to the physical symmetry between $\mathbf{k}$ and $-\mathbf{k}$, we also have that $u_{\mathbf{k}} = u_{-\mathbf{k}}$ and $v_{\mathbf{k}} = v_{-\mathbf{k}}$. Note that each quasiparticle operator carries a definite wavevector and spin. Thus one operator creates an electron and the other operator creates a hole with the same quantum numbers. The Bogoliubov quasiparticle would appear to be a mix of electron and hole, and it is tempting to call it a resonance between them. While this may provide some physical insight in a different context, in our case we simply use the transformation as a means to diagonalize (14). To proceed we look at the the Hamiltonian in this basis by substituting the transformations (15) and (16) into (14). This calculation is straightforward but lengthy and uninsightful, and thus we will skip to the final result. In the interest of completeness, a detailed line-by-line derivation of this step is given in lecture notes

prepared by Mahajan and Singh [18].

$$H' = \sum_{\mathbf{k}} \Big\{ \xi(\mathbf{k}) \left[(u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2)(\gamma^\dagger_{\mathbf{k}\uparrow}\gamma_{\mathbf{k}\uparrow} + \gamma^\dagger_{-\mathbf{k}\downarrow}\gamma_{-\mathbf{k}\downarrow}) + 2|v_{\mathbf{k}}|^2 + 2u^\star_{\mathbf{k}}v^\star_{\mathbf{k}}\gamma_{-\mathbf{k}\downarrow}\gamma_{\mathbf{k}\uparrow} + 2u_{\mathbf{k}}v_{\mathbf{k}}\gamma^\dagger_{\mathbf{k}\uparrow}\gamma^\dagger_{-\mathbf{k}\uparrow} \right] + \left[(\Delta_{\mathbf{k}}u_{\mathbf{k}}v^\star_{\mathbf{k}} + \Delta^\star_{\mathbf{k}}u^\star_{\mathbf{k}}v_{\mathbf{k}})(\gamma^\dagger_{\mathbf{k}\uparrow}\gamma_{\mathbf{k}\uparrow} + \gamma^\dagger_{-\mathbf{k}\downarrow}\gamma_{-\mathbf{k}\downarrow} - 1) + (\Delta_{\mathbf{k}}v^{\star 2}_{\mathbf{k}} - \Delta^\star_{\mathbf{k}}u^{\star 2}_{\mathbf{k}})\gamma_{-\mathbf{k}\downarrow}\gamma_{\mathbf{k}\uparrow} + (\Delta^\star_{\mathbf{k}}v^2_{\mathbf{k}} - \Delta_{\mathbf{k}}u^2_{\mathbf{k}})\gamma^\dagger_{\mathbf{k}\uparrow}\gamma^\dagger_{-\mathbf{k}\downarrow} + \Delta_{\mathbf{k}}b^\star_{\mathbf{k}} \right] \Big\} \tag{17}$$

To complete the diagonalization, we must choose $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ such that the coefficients of $\gamma_{-\mathbf{k}\downarrow}\gamma_{\mathbf{k}\uparrow}$ and $\gamma^\dagger_{\mathbf{k}\uparrow}\gamma^\dagger_{-\mathbf{k}\downarrow}$ vanish so that the Hamiltonian only contains the number operators $\gamma^\dagger_{\mathbf{k}\uparrow}\gamma_{\mathbf{k}\uparrow}$ and $\gamma^\dagger_{-\mathbf{k}\downarrow}\gamma_{-\mathbf{k}\downarrow}$. By inspection, this is equivalent to the condition

$$2\xi_{\mathbf{k}}u_{\mathbf{k}}v_{\mathbf{k}} + \Delta^\star_{\mathbf{k}}v^2_{\mathbf{k}} - \Delta_{\mathbf{k}}u^2_{\mathbf{k}} = 0 \tag{18}$$

Multiplying through by $\Delta^\star_{\mathbf{k}}/u^2_{\mathbf{k}}$ we obtain a quadratic equation

$$\left(\frac{\Delta^\star_{\mathbf{k}}v_{\mathbf{k}}}{u_{\mathbf{k}}}\right)^2 + 2\xi_{\mathbf{k}}\left(\frac{\Delta^\star_{\mathbf{k}}v_{\mathbf{k}}}{u_{\mathbf{k}}}\right) - |\Delta_{\mathbf{k}}|^2 = 0 \tag{19}$$

whose positive-energy solution is

$$\frac{\Delta^\star_{\mathbf{k}}v_{\mathbf{k}}}{u_{\mathbf{k}}} = \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2} - \xi_{\mathbf{k}} \equiv E_{\mathbf{k}} - \xi_{\mathbf{k}} \tag{20}$$

Squaring both sides then taking the square root we obtain the absolute value of the term on the left hand side

$$\left|\frac{\Delta^\star_{\mathbf{k}}v_{\mathbf{k}}}{u_{\mathbf{k}}}\right| = E_{\mathbf{k}} - \xi_{\mathbf{k}} \tag{21}$$

Recalling the normalization condition $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$, we can solve for the transformation coefficients

$$|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right), \quad |v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \tag{22}$$

from which we see that

$$2u_{\mathbf{k}}v_{\mathbf{k}} = \frac{\Delta_{\mathbf{k}}}{E_{\mathbf{k}}} \tag{23}$$

This lets us express (8) using (15) and (16) as

$$b_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}\rangle = \frac{1}{2}\left(\frac{\Delta_{\mathbf{k}}}{E_{\mathbf{k}}}\right)\left(1 - \langle n^\gamma_{\mathbf{k}\uparrow}\rangle - \langle n^\gamma_{\mathbf{k}\downarrow}\rangle\right) \approx \frac{1}{2}\left(\frac{\Delta_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \tag{24}$$

which finally lets us write the energy gap (12) as

$$\Delta_{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{k}'}\langle \mathbf{k}|V|\mathbf{k}'\rangle \frac{\Delta_{\mathbf{k}'}}{(\Delta^2_{\mathbf{k}'} + \xi^2_{\mathbf{k}'})^{\frac{1}{2}}} \tag{25}$$

giving the famous BCS gap equation we set out to obtain.

III. CALCULATING THE ENERGY GAP IN NUCLEAR MATTER

We now proceed to calculate the energy gap in nuclear matter following chapter 43 of Fetter and Walecka [19]. In applying the gap equation (25) to nuclear matter, the matrix element $\langle \mathbf{k}|V|\mathbf{k}'\rangle$ must actually be replaced by an anti-symmetrized matrix element describing the two-body interaction $\langle \mathbf{k}-\mathbf{k}|\mathcal{V}|\mathbf{k}'-\mathbf{k}'\rangle$ [20] where

$$\langle \mathbf{pq}|\mathcal{V}|\mathbf{p}'\mathbf{q}'\rangle = \frac{1}{4}\left(\langle \mathbf{pq}|V|\mathbf{p}'\mathbf{q}'\rangle - \langle \mathbf{pq}|V|\mathbf{q}'\mathbf{p}'\rangle - \langle \mathbf{qp}|V|\mathbf{p}'\mathbf{q}'\rangle + \langle \mathbf{qp}|V|\mathbf{q}'\mathbf{p}'\rangle\right) \tag{26}$$

and

$$\langle \mathbf{pq}|V|\mathbf{p'q'}\rangle = \int \Psi^{\star}_{\mathbf{p}}(\mathbf{r})\Psi^{\star}_{\mathbf{q}}(\mathbf{r'})V(\mathbf{r},\mathbf{r'})\Psi_{\mathbf{p'}}(\mathbf{r})\Psi_{\mathbf{q'}}(\mathbf{r'})d\mathbf{r}d\mathbf{r'} \tag{27}$$

so $V(\mathbf{r},\mathbf{r'})$ is the two-body interaction and $\Psi_{\mathbf{p}}(\mathbf{r})$ are plane waves with momentum $\mathbf{p}$. This general form of the interaction is mostly formal in our cases. $\mathbf{k}$ and $-\mathbf{k}$ are related by symmetry and so we may as well write our matrix element as

$$\langle \mathbf{k}-\mathbf{k}|\mathcal{V}|\mathbf{k'}-\mathbf{k'}\rangle \equiv \langle \mathbf{k}|\mathcal{V}|\mathbf{k'}\rangle \tag{28}$$

We may write the gap equation (25) in integral form if we take $\mathbf{k}$ to be continuous as opposed to discrete.

$$\Delta(\mathbf{k}) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\langle \mathbf{k}|\mathcal{V}|\mathbf{k'}\rangle\,\frac{\Delta(\mathbf{k'})}{[\Delta^2(\mathbf{k'})+\xi^2(\mathbf{k'})]^{\frac{1}{2}}} \tag{29}$$

This is a nonlinear integral equation and to obtain an explicit solution, we'll make some assumptions and approximations. We shall restrict pairing to like particles only, that is *pp* or *nn* only, so that BCS theory actually applies and we shall assume that $\mathcal{V}$ is independent of spin and isospin. We will assume that the single-particle excitation energies ξ_k measured relative to the chemical potential μ are written in the effective-mass approximation

$$\xi(\mathbf{k}) = \frac{\hbar^2}{2m^{\star}}(k^2-k_F^2) \tag{30}$$

Also, since the resulting gap Δ is much smaller than the Fermi energy ϵ_F, the integrand will be sharply peaked near the Fermi surface $\xi = 0$. We can thus approximate $\mathbf{k} \approx \mathbf{k'} \approx \mathbf{k}_F$ and

$$\Delta(\mathbf{k}) \approx \Delta(\mathbf{k'}) \equiv \Delta \tag{31}$$

where we shall use Δ from now on for brevity. The gap equation (29) now becomes, after dividing both sides by Δ,

$$1 = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\frac{\langle \mathbf{k}_F|\mathcal{V}|\mathbf{k}\rangle}{\{\Delta^2+[\hbar^2(k^2-k_F^2)/2m^{\star}]^2\}^{\frac{1}{2}}} \tag{32}$$

We can make a change of variables to the new dimensionless $\bar{\Delta}$ and K where

$$\bar{\Delta} = \frac{\Delta}{\hbar^2k_F^2/2m^{\star}} = \frac{\Delta}{\epsilon_F^{\star}}, \quad K = \frac{k}{k_F} \tag{33}$$

which lets us write the denominator of (32) as

$$\left\{\Delta^2 + \left[\frac{\hbar^2(k^2-k_F^2)}{2m^{\star}}\right]^2\right\}^{\frac{1}{2}} = \frac{\hbar^2k_F^2}{2m^{\star}}[\bar{\Delta}^2+(K^2-1)^2]^{\frac{1}{2}} \tag{34}$$

and so (32) becomes

$$\begin{aligned} 1 &= \frac{m^{\star}}{\hbar^2k_F^2}\int \frac{d^3kd^3x}{(2\pi)^3}\,\frac{e^{-i\mathbf{k}_F\cdot\mathbf{x}}\mathcal{V}(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}}{[\bar{\Delta}^2+(K^2-1)^2]^{\frac{1}{2}}} \\ &= \frac{m^{\star}}{\hbar^2k_F^2}\int d^3x\mathcal{V}(\mathbf{x})e^{-i\mathbf{k}_F\cdot\mathbf{x}}\int \frac{d^3k}{(2\pi)^3}\,\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{[\bar{\Delta}^2+(K^2-1)^2]^{\frac{1}{2}}} \end{aligned} \tag{35}$$

We now make use of the following integral identity given on page 31 of Zee [21] which we used as an intermediate step for a derivation on an assignment

$$\int \frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{2}{x}\int_0^{\infty}\frac{kdk}{(2\pi)^2}\sin(kx) \tag{36}$$

such that we can express the d^3x integral as

$$\int d^3x\,\mathcal{V}(\mathbf{x})e^{i\mathbf{k}_F\cdot\mathbf{x}} = \frac{2(2\pi)}{k_F}\int_0^{\infty}dx\,x\mathcal{V}(\mathbf{x})\sin(k_Fx) \tag{37}$$

while the d^3k integral becomes

$$\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{[\bar{\Delta}^2+(K^2-1)^2]^{\frac{1}{2}}} = \frac{2}{x}\int_0^\infty \frac{kdk}{(2\pi)^2}\frac{\sin(kx)}{[\bar{\Delta}^2+(K^2-1)^2]^{\frac{1}{2}}}$$
$$= \frac{2k_F^2}{(2\pi)^2 x}\int_0^\infty KdK\,\frac{\sin[K(k_Fx)]}{[\bar{\Delta}^2+(K^2-1)^2]^{\frac{1}{2}}} \tag{38}$$

turning (35) into

$$1 = \frac{2m^\star}{\pi\hbar^2k_F^2}\int_0^\infty dx\,\mathcal{V}(\mathbf{x})k_F\sin(k_Fx)\int_0^\infty KdK\,\frac{\sin[K(k_Fx)]}{[\bar{\Delta}^2+(K^2-1)^2]^{\frac{1}{2}}} \tag{39}$$

We are interested in the case that $\bar{\Delta}\ll 0$ as we know the energy gap to be tiny where the second integral can be evaluated as

$$\lim_{\bar{\Delta}\to 0}\int_0^\infty KdK\,\frac{\sin[K(k_Fx)]}{[\bar{\Delta}^2+(K^2-1)^2]^{\frac{1}{2}}} = \left[\ln\left(\frac{8}{\bar{\Delta}}\right)-\int_0^{2k_Fx}\frac{d\lambda}{\lambda}(1-\cos\lambda)\right]\sin(k_Fx)+\left(\int_{2k_Fx}^\infty\frac{d\lambda}{\lambda}\sin\lambda\right)\cos(k_Fx) \tag{40}$$

as discussed on page 384 of Fetter and Walecka [19]. It is left as an exercise in the textbook and is evaluated in appendix A by Kennedy, Wilets, and Henley [10]. We have not done so yet but we will assume a finite square-well potential for $\mathcal{V}$(x) which has a finite range. The finite range of the potential ensures that k_Fx is bounded and so the dominant behavior of this integral is contained in the first term, allowing us to evaluate the momentum integral as

$$\lim_{\bar{\Delta}\to 0}\int_0^\infty KdK\,\frac{\sin[K(k_Fx)]}{[\bar{\Delta}^2+(K^2-1)^2]^{\frac{1}{2}}} = \sin(k_Fx)\ln\left(\frac{8}{\bar{\Delta}}\right) \tag{41}$$

Substituting this into (39) we finally arrive at a tangible form for the gap equation

$$1 = \frac{2m^\star}{\pi\hbar^2k_F^2}\ln\left(\frac{8}{\bar{\Delta}}\right)\int_0^\infty dx\,k_F\mathcal{V}(\mathbf{x})\sin^2(k_Fx) = \frac{2m^\star}{\pi\hbar^2k_F^2}\ln\left(\frac{8}{\bar{\Delta}}\right)\langle k_F|\mathcal{V}|\varphi_{k_F}\rangle \tag{42}$$

where we introduced for convenience that

$$\langle k_F|\mathcal{V}|\varphi_{k_F}\rangle = \int_0^\infty dx\,k_F\mathcal{V}(\mathbf{x})\sin^2(k_Fx) \tag{43}$$

We can invert (42) to obtain

$$\bar{\Delta} = 8\exp\left(-\frac{\pi\hbar^2k_F^2/2m^\star}{\langle k_F|\mathcal{V}|\varphi_{k_F}\rangle}\right) \tag{44}$$

or back in SI units0as

$$\Delta = 8\frac{\hbar^2k_F^2}{2m^\star}\exp\left(-\frac{\pi\hbar^2k_F^2/2m^\star}{\langle k_F|\mathcal{V}|\varphi_{k_F}\rangle}\right) \tag{45}$$

We now choose a crude nonsingular square-well potential fit to low-energy $^1S_0$ scattering for the two-body nuclear potential as given in Fig. 41.2 of Fetter and Walecka [19]

$$\mathcal{V}(\mathbf{x}) = \begin{cases} -V_0 & x<d \\ 0 & x\geq d \end{cases} \tag{46}$$

where $d = 2.7$ F and $V_0 = 14$ MeV. We can now evaluate (43) exactly as

$$\langle k_F|\mathcal{V}|\varphi_{k_F}\rangle = -V_0\int_0^d dx\,k_F\sin^2(k_Fx) = V_0\left(\frac{\sin(2k_Fd)}{4}-\frac{k_Fd}{2}\right) \tag{47}$$

IV. CONCLUSION

Fetter and Walecka [19] give the equilibrium density of nuclear matter to be $k_F \approx 1.42$ F^{-1} and the effective mass to be $m^\star = 0.65m$. Using this we can plot the gap Δ as a function of the nuclear density and effective mass in Fig. 2. As expected, we underestimate the gap especially in the region near $k_F = 1.42$ F^{-1} which we were interested in. This makes sense in hindsight as we've attempted to calculate the gap in infinite nuclear matter when it arises due to nucleon pairing near the surface of the nucleus where the density is much lower. While it is tempting to discuss the minimum seen in Fig. 2 (left), regular nuclear matter does not get that dense and it is an artifact of our simple potential fit failing for denser nuclei. This shows a clear strong dependence of the gap on the nuclear density, but not so much on the effective mass as it can only take on a smaller range of values. Emery and Sessler [6] have used the Bethe-Goldstone equation to obtain much more realistic values of $\langle k_F|\mathcal{V}|\varphi_{k_F}\rangle$. Still, they predict a similar dependence for the nuclear density (Fig. 2, left) but a more sensitive exponential dependence on $m^\star/m$.

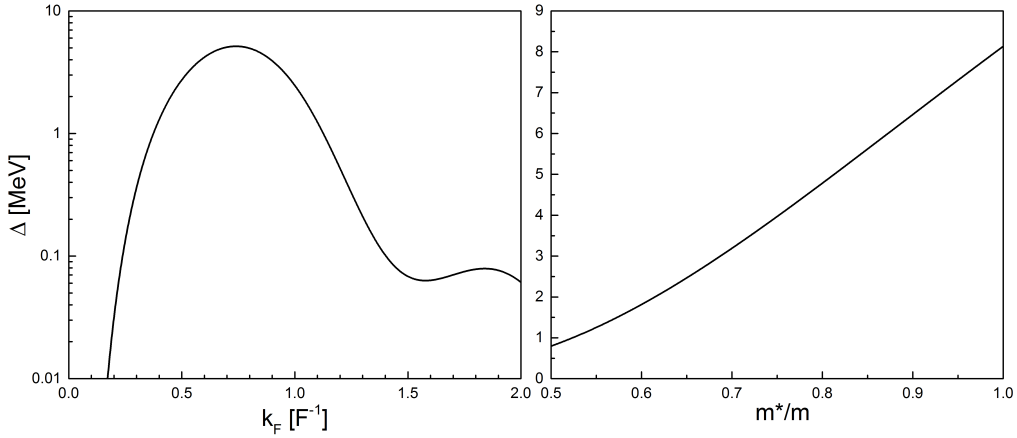

FIG. 2. The energy gap in nuclear matter as a function of nuclear density with $m^\star/m = 0.65$ (left) and as a function of the effective mass at the Fermi surface with $k_F = 1.0$ F^{-1} (right).

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